Al-FARABI KAZAKH NATIONAL UNIVERSITY

**Faculty of Mechanics and Mathematics**

**Department of Mathematical and Computer Modeling**

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|  | **Approve**  **at a faculty academic council meeting**  **The transactions No , 2015**  Dean of Faculty M. A. Bektemesov |

# SYLLABUS

## Numerical Methods ODE & PDE (NM ODE&PDE)

**2 course, English,** **a** **second half-year, 3 credits = 2 lecture + 1 laboratory**

**Professor: Kanat Shakenov,**

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**Goals and objectives.** I believe that modern computers can play a similar role in mathematics. This course presents the innovative approach that numerical methods should be considered as a practical laboratory for undergraduate mathematics courses. I think this is innovative because it is not the state of affairs we currently encounter. On the one hand, first-, second- and third-year students in mathematics, science, and engineering learn introductory mathematical concepts without making appropriate use of computer technology. On the other hand, upper-level courses on numerical methods put their emphases on specific topics such as computational algorithms, error analysis, convergence and stability, and coding and debugging procedures, with only passing references to the mathematical background, which is generally assumed to be known and understood beforehand.

Differential equations describe the evolution of various quantities in many physical, chemical, biological, and economical problems. When the quantities depend on one variable, such as time, and do not depend on other variables, such as spatial coordinates, they may satisfy certain relations between the function and its first- and higher-order derivatives called ***ordinary differential equations****,* or simple***ODEs****.* Depending on the applied context of the problem, the only independent variable can have different meanings, for example, time, a space coordinate, a parameter. When the ODEs are used to model the evolution dynamics of a physical system starting with a given initial state, it makes sense to call the independent variable the *time variable,* as we do in this lecture. Two mathematical problems are associated with ordinary differential equations depending on whether the system of ODEs is supplemented by an ***initial*** value of the function at a single instance of the independent variable or its ***boundary*** values at distinct instances of the independent variable. This lectures covers existence and properties of solutions of the initial-value problems for ODEs, algorithms and errors of their numerical approximations, and an interplay between the numerical algorithms and their convergence, stability, and robustness.

***Boundary-value Problems for ODEs and PDEs (Partial Differential Equation).*** Whereas solutions of initial-value problems for ordinary differential equations (ODEs) exist whenever the vector field for the differential equation is sufficiently smooth, boundary-value problems may have no solutions if the boundary conditions are inconsistent. Nevertheless, many boundary-value problems arise in the context of physical and engineering problems where the existence of solutions can be complicated, it is legitimate to develop numerical approximations and graphical visualizations of the solutions before analyzing the properties of the given boundary-value problem, consistent with general strategy adopted in our numerical laboratory. Many specific results on solutions of boundary-value problems for ODEs and PDEs cannot be formulated as general theorems. Moreover, no general numerical recipe is available for numerical solutions of nonlinear differential equations. Experience and knowledge of various numerical routines are valuable assets in the design of new numerical algorithms for solutions of boundary-value problems. This lecture covers the simplest numerical approximations of solutions of boundary-value problems associated with ODEs and PDEs, as well as convergence, stability, and robustness of the numerical algorithms.

Trans-property: Algebra, Geometry, Mathematical analysis, ODE, PDE, Functional analysis, Integral equations, Computer science, Discrete mathematics, Numerical Methods Algebra and Analysis

Post-property: Monte Carlo Methods, Mathematical Modeling, Computer Modeling, Numerical Mathematics, Numerical Fluid Mechanics, Hydrodynamics, Filtration Process.

**The structure of the course.**

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| --- | --- | --- | --- | --- |
| **Weeks** | **Name of subject (theme)** | **duration** | **Students self-instruction (SSI) by subject** | |
| Module 1. Computational Methods Solution of the Initial-Value Problemsfor ODEs | | | | |
| 1 | **Lecture 1-2.** Approximations of the normalized spaces. Outer and internal approximations. Discrepancy (disparity, residual), residual function. Approximation error. Discrete residual. Stability, stable operator. Convergence, discrete convergence.  Approximations of Solutions. Initial-Value ODE Problem. Euler’s method and Convergence of Euler’s method. Heun’s and modified Euler’s methods. Stability of Euler’s method.  **Laboratory 1-2.** Theorem 1(Initial-Value ODE Problem). Theorem 2. Exercise 1. Exercise 2. Exercise 3. Numerical approximation of the Euler’s method. Approximation error and stability of Euler’s method. | 2  2 | | SSI-1-2 Approximations of Solutions. Initial-Value ODE Problem. Local and global truncation errors for the explicit Euler’s method for the ODE. |
| 2 | **Lecture 3-4.** Single-Step Runge-Kutta Solvers.Problem 2. Explicit th Order Runge-Kutta Methods. First- and second-order Runge-Kutta methods. Stability of second-order methods.  **Laboratory 3-4.** Heun’s and modified Euler’s methods. Stability of Heun’s and modified Euler’s method. Exercise 4. | 2  2 | | **SSI-3-4**  Find numerical approximations of the solution of the ODE problem using Heun’s third-order, and fourth-order Runge-Kutta methods and compare the time when numerical instabilities of iterations are developed in each method. |
| 3 | **Lecture 5-6.** Higher-order Runge-Kutta methods. Third-order and Fourth-order Runge-Kutta methods. Approximations of the Third-order and Fourth-order Runge-Kutta methods. Convergence and stability of the Third-order and Fourth-order Runge-Kutta methods.  **Laboratory 5-6.**  Exercise 5, Exercise 6. Adaptive Single-Step Solvers. | 2  2 | | **SSI-5-6**  Exercise 7, Exercise 8, Exercise 9, Exercise 10. |
| 4 | **Lecture 7-8.** Multistep Adams Solvers. Adams-Bashforth methods. Explicit and Implicit Multistep Adams methods. Problem 1. Explicit -Order Adams ODE Solver. Convergence and stability. Adams-Moulton methods. Implicit -Order Adams ODE Solver.  **Laboratory 7-8.** Exercise 11.Derive the Adams methods from integration of the integral equation. Numerical solution of the ODE with the second-order explicit Adams method. | 2  2 | | **SSI-7-8**  Derive the implicit Adams methods from integration of the integral equation. Exercise 12, Exercise 13. |
| 5 | **Lecture 9-10.** Predictors and Correctors. Convergence and Stability. Time step adjustment.  **Laboratory 9-10.** Derive the fourth-order predictor-corrector pair and the fourth-order Runge-Kutta method.Exercise 15, Exercise 16. | 2  2 | | **SSI-9-10**  Exercise 14. |
| 6 | **Lecture 11-12.** Implicit Methods for Stiff Differential Equations.Stiff ODE system. Solution Stiff ODE systems by iterations of the explicit single-step or multistep ODE solvers, such as the Runge-Kutta and Adams methods. Implicit method.  **Laboratory 11-12.** Numerical solution of the ODE system with the explicit Euler and Heun methods. Exercise 17, Exercise 18. | 2  2 | | **SSI-11-12**  Numerical solution of the ODE system with the explicit Euler and Heun methods. |
| Module 2. Computational Methods Solution of the Boundary-Value Problems **for ODEs** | | | | |
| 7 | Lecture 13-14. William Edmund Milne Method. Computational Methods Solution of the Boundary-Value Problems for ODEs. Finite-Difference Methods for ODEs. Problem 1. Boundary-Value Second-Order ODE Problem. Dirichlet boundary conditions. Neumann boundary conditions. Theorem 1. Stability. **Laboratory 13-14.** Approximation errors. The error of the numerical approximation in the finite-difference solution of the linear boundary-value problem. Stability. | 2  2 | | **SSI-13-14**  Numerical error of the finite-difference solution of the linear equation. Stability. |
|  | **Total Control (TC) No.1 (Weeks 1 – 7 )** | 2 | |  |
| 8 | Lecture 15-16. Sweep Method Solution of the Boundary-Value Second-Order ODE Problems. Approximation Errors. Stability. Galerkin's method. Theorem 1. Complete orthonormal basis. Least-squares method. Shooting Methods for ODEs.Laboratory 15-16. Mixed Boundary-Value Second-Order ODE Problem. Approximation errors. Rising of exactness. Nonlinear ODEs. Exercise 1. | 2  2 | | **SSI-15-16**  The Approximation Errors of Exactness rising. Nonlinear ODEs. Exercise 1. |
| 9 | Lecture 17-18. Shooting Methods for ODEs. Linear shooting method. Newton-Raphson method. Secant method.Laboratory 17-18. Exercise 2, Exercise 3. The modification of the shooting method for the Bessel equation. | 2  2 | | **SSI-17-18**  Sturm-Liouville Theory. Evans function. |
| 10 | Lecture 19-20. Shooting Methods for ODEs. Linear shooting method. Newton-Raphson method. Secant method.Laboratory 19-20. Exercise 2, Exercise 3. The modification of the shooting method for the Bessel equation. | 2  2 | | **SSI-19-20**  Sturm-Liouville Theory. Evans function. |
| Module 3. Computational Methods Solution of the Boundary-Value Problems **for PDEs** | | | | |
| 11 | Lecture 21-22. Finite-Difference Methods for Parabolic PDEs. Problem 1. Parabolic PDE Problem. Explicit method. The Local error of the Explicit methods. Implicit method. The Global truncation error of the Implicit methods. Crank-Nicolson method. The Global truncation error of the Crank-Nicolson methods. Stability of iterations. Convergence of iterations. The discrete Fourier analysis of stability. The conditional stability of the explicit method. Analysis of convergence and stability of the finite-difference methods by discrete Fourier method.Laboratory 21-22. Exercise 4. Approximation errors and Stability of iterations. Convergence of iterations. Analysis of convergence and stability of the finite-difference methods by discrete Fourier method. | 2  2 | | **SSI-21-22** Stability of iterations. The discrete Fourier analysis of stability. The conditional stability of the explicit method. Analysis of convergence and stability of the finite-difference methods by discrete Fourier method. |
| 12 | Lecture 23-24. Finite-Difference Methods for Hyperbolic PDEs. Problem 2. Hyperbolic PDE Problem. Explicit method. Implicit method. Courant number. Initial time step. Convergence and Stability. Conditionally stable. Inconditionally stable.The discrete Fourier analysis of stability. Analysis of convergence and stability of the finite-difference methods by discrete Fourier method. Exact finite-difference solution. The D’Alambert form.Laboratory 23-24. Exercise 5. Approximation errors and Stability of iterations. Analysis of convergence and stability of the finite-difference methods by discrete Fourier method. | 2  2 | | **SSI-23-24** Stability of iterations. Convergence of iterations. The discrete Fourier analysis of stability. Analysis of convergence and stability of the finite-difference methods by discrete Fourier method. |
| 13 | Lecture 25-26. Finite-Difference Methods for Elliptic PDEs. Problem 3. Elliptic PDE Problem. Dirichlet boundary-values problem for Poisson equation. Direct method. Theorem 2. Relaxation methods. Underrelaxation method. Overrelaxation method. Stability of relaxation methods. Alternative direction method. Nonlinear elliptic problem.Laboratory 25-26. Exercise 6. Jacobi iterations. Gauss-Seidel iterations. Exercise 7. Exercise 8. Alternative direction method. Nonlinear elliptic problem. | 2  2 | | **SSI-25-26** Stability of iterations. Alternative direction method. Nonlinear elliptic problem. Exercise 7. Exercise 8. |
| 14 | Lecture 27-28. Variational Methods for PDEs. Ritz method. Theorem 1. Characterization of the Ritz methods. Definition 1. Theorem 2.Laboratory 27-28. Classification of the Conditions. Natural conditions (Neumann condition). General conditions (Dirichlet condition). | 2  2 | | **SSI-27-28** Penalty method. |
| 15 | Lecture 29-30. Solution of the Diffusion difference equation by Ritz methods. Sweep Method Solution of Diffusion difference equation. Stability.Laboratory 29-30. Numerical solution of Diffusion difference equation. Stability condition. | 2  2 | | **SSI-29-30** Capital property of the Variational Methods. |
|  | **Total Control (TC) No.2 (Weeks 8 – 15 )** | 2 | |  |

**References**

**Basic:**

1. K.K. Shakenov. Numerical Mathematics Methods. Lecture Courses. Print S. Almaty, 2009. (In Kazakh).
2. Matheus Grasselli, Dmitry Pelinovsky. Numerical Mathematics. Narosa Publishing House. India. 2009.
3. Walter Gautschi. Numerical Analysis. Second Edition. Birkhäuser/ 2012/
4. Olaf Steinbach. Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer. 2008.
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6. Ioannis K. Argyros, Said Hilout. Computational Methods in Nonlinear Analysis. Efficient Algorithms, Fixed Point Theory and Applications/ 2013.
7. George E. Forsythe, Wolfgang R. Wasow. FINITE-DIFFERENCE METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS. JOHN WILEY & SONS, INC. NEW YORK – LONDON, 1959.
8. Robert D. Richtmyer, K.W. Morton. DIFFERENCE METHODS FOR INITIAL – VALUE PROBLEMS. Second Edition. INTERSCIENCE PUBLISHERS a division of John Wiley & Sons. NEW YORK LONDON SYDNEY, 1967.
9. JEAN-PIERRE AUBIN. APPROXIMATION OF ELLIPTIC BOUNDARY-VALUE PROBLEMS. WILEY-INTERCSIENCE a Division of John Wiley & Sons, Inc. New York London Sydney Toronto, 1972.
10. C. A. J. Fletcher. COMPUTATIONAL GALERKIN METHODS. Springer-Verlag. New York Berlin Heidelberg Tokyo, 1984.

**Additional:**

1. R. W. HAMMING. NUMERICAL METHODS FOR SCIENTISTS AND ENGINEERS. MC

GRAW-HILL BOOK COMPANY. INC. NEW YORK, SAN FRANCISCO, TORONTO, LONDON,

1962.

1. WILLIAM EDMUND MILNE. NUMERICAL CALCULUS. Approximations, Interpolation, Finite

Differences, Numerical Integration and Curve Fitting. PRINCETON UNIVERSITY PRESS.

PRINCETON, NEW JERSEY, 1949.

1. KAISER S. KUNZ. NUMERICAL ANALYSIS. McGraw-Hill Book Company, Inc. NEW YORK

TORONTO LONDON, 1957.

1. Roger TEMAM. NAVIER – STOKES EQUATIONS. Theory and numerical analysis. Revised

Edition. North-Holland Publishing Company. Amsterdam New York Oxford. 1979.

**Control Test: twice.**

**SSI***:* **a few times.**

**Criterion of grade of the knowledge, marks in percent**

|  |  |
| --- | --- |
| *Lecture* | *30 +*  *30* |
| *SSI – theory* |
| *SSI – practice(seminar)* | *30* |
| *Total written examination* | *90* |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Weeks** | **Lecture** | | **Seminar** | | **SSI** | | **TC** | **TOTAL** |
|  | **No.** | **Mark** | **No.** | **Mark** | **No.** | **Mark** |  | **Mark** |
| 1 | 1 | 0,5 | 1 | 2 | 1 | 0,5 |  | 3 |
| 2 | 2 | 0,5 | 2 | 2 | 2 | 0,5 |  | 3 |
| 3 | 3 | 0,5 | 3 | 2 | 3 | 0,5 |  | 4 |
| 4 | 4 | 0,5 | 4 | 4 | 4 | 0,5 |  | 5 |
| 5 | 5 | 0,5 | 5 | 5 | 5 | 0,5 |  | 6 |
| 6 | 6 | 0,5 | 6 | 3 | 6 | 0,5 |  | 4 |
| 7 | 7 | **-** | 7 | 3 | 7 | **-** | 3 | 6 |
| **Total: Weeks 1-7** |  | **3** |  | **21** |  | **3** | **3** | **30** |
| 8 | 8 | 0,5 | 8 | 2 | 8 | 0,5 |  | 3 |
| 9 | 9 | 0,5 | 9 | 4 | 9 | 0,5 |  | 5 |
| 10 | 10 | 0,5 | 10 | 3 | 10 | 0,5 |  | 4 |
| 11 | 11 | 0,5 | 11 | 3 | 11 | 0,5 |  | 4 |
| 12 | 12 | 0,5 | 12 | 2 | 12 | 0,5 |  | 3 |
| 13 | 13 | 0,5 | 12 | 1 | 13 | 0,5 |  | 2 |
| 14 | 14 | - | 12 | 3 | 14 | - |  | 3 |
| 15 | 15 | - | 12 | 3 | 15 | - | 3 | 6 |
| **Total: Weeks 8-15** |  | **3** |  | **21** |  | **3** | **3** | **30** |
| **Total: Weeks 1-15** |  | **6** |  | **42** |  | **6** | **6** | **60** |

**The scale of mark of knowledge:**

|  |  |  |  |
| --- | --- | --- | --- |
| **Letter symbol of mark** | **Digital of mark (GPA)** | **Mark (percent)** | **Mark on tradition system** |
| A | 4 | 95-100 | “excellent” |
| A- | 3,67 | 90-94 |
| B+ | 3,33 | 85-89 | “good” |
| B | 3 | 80-84 |
| B- | 2,67 | 75-79 |
| C+ | 2,33 | 70-74 | “satisfactory” |
| C | 2 | 65-69 |
| C- | 1,67 | 60-64 |
| D+ | 1,33 | 55-59 |
| D | 1 | 50-54 |
| F | - | 0-49 | “unsatisfactory” |
| I | - | - | “Incomplete discipline” |
| W | - | - | “Renunciation of discipline” |
| AW | - | - | “Deduction off discipline” |
| AU | - | - | “To take a discipline” |
| P/NP (Pass / No Pass) | - | 65-100/0-64 | “Pass / No Pass” |

Sitting of the chair consideration

Protocol No. , , 2015

**Acting as chief of chair M&C M,**

PhD, docent \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Dauren Zhakebaev

**Lecturer**

Doctor of Science, Professor \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Кanat Shakenov